Axisymmetric Parabolic Loading of Anisotropic Halfspace

By George Gazetas,[1] M. ASCE

Introduction

In a companion technical paper (2), the writer studied the effect of soil cross-anisotropy on stress and displacement distributions in a homogeneous elastic halfspace subjected to normal axisymmetric surface loading distributed in the

---

[1] Assoc. Prof. of Civ. Engrg., Rensselaer Polytechnic Inst., Troy, N.Y., 12181.

Note.—Discussion open until September 1, 1982. To extend the closing date one month, a written request must be filed with the Manager of Technical and Professional Publications, ASCE. Manuscript was submitted for review for possible publication on December 30, 1980. This paper is part of the Journal of the Geotechnical Engineering Division, Proceedings of the American Society of Civil Engineers, ©ASCE, Vol. 108, No. GT4, April, 1982. ISSN 0093-6405/82/0004-0654/$01.00.

form of a convex parabola of revolution. The study was based on an analytical solution which was only briefly outlined in that paper (2). This technical note:

1. Presents the complete formulation and solution of that boundary value problem (hereafter referred to as 'frictionless' loading case).
2. Outlines an analytical solution to the problem of 'adhesive' parabolic loading of a cross-anisotropic half-space; such a loading may be appropriate in case of a raft foundation that is flexible in bending but has a very large in-plane stiffness thus preventing any horizontal displacements from occurring within the contact area.
3. Graphically shows the effect of soil anisotropy on vertical and radial stresses and displacements under both 'frictionless' and 'adhesive' loading conditions.

General Solution for Frictionless Parabolic Loading

A cross-anisotropic medium with a vertical axis of material symmetry is characterized by five independent elastic material constants: a vertical and a horizontal Young's modulus, E_V and E_H, a shear modulus on vertical planes, G_{VH}, a Poisson's ratio for horizontal due to vertical strain, ν_{VH}, and a Poisson's ratio for horizontal due to horizontal strain, ν_{HH} (2,6). Under conditions of axial symmetry, and in the absence of body forces and torsional deformations, it has been shown (6) that the distribution of displacements and stresses can be expressed in terms of a potential function, $\Phi = \Phi(r, z)$, whose zero-order Hankel transform is

$$\bar{\Phi}(\xi, z) = \int_o^\infty r\,\Phi(r, z)\,J_o(\xi r)\,dr \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which J_o = the first-kind, zero-order Bessel function, and this satisfies the following ordinary differential equation:

$$\left(\frac{d^2}{dz^2} - s_1^2\xi^2\right)\left(\frac{d^2}{dz^2} - s_2^2\xi^2\right)\bar{\Phi}(\xi, z) = o \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The general solution of Eq. 2 takes the form:

$$\bar{\Phi}(\xi, z) = A(\xi)\exp(-\xi s_1 z) + B(\xi)\exp(-\xi s_2 z) \quad \ldots\ldots\ldots\ldots \quad (3)$$

in which s_1 and s_2 = material constants given in terms of E_H, E_V, ν_{VH}, ν_{HH}, and G_{VH} in Appendix I, while the constants of integration, $A(\xi)$ and $B(\xi)$, are evaluated from the boundary conditions of the problem. At the surface, $z = o$, the shear stresses, τ_{rz}, must be everywhere zero (frictionless loading), while the normal stresses, σ_z, are equal to the applied normal tractions that are distributed as a convex parabola of revolution: $p = p_o(1 - \rho^2)$, with $\rho = r/R \leq 1$. If we introduce the zero-order Hankel transform, $\bar{\sigma}_z(\xi, z)$ of σ_z, and the first-order Hankel transform, $\bar{\bar{\tau}}_{rz}(\xi, z)$ of τ_{rz}, the two boundary conditions can be written as

$$\bar{\sigma}_z(\xi, o) = \bar{p}(\xi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

$$\bar{\bar{\tau}}_{rz}(\xi, o) = o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

in which $\bar{p}(\xi)$ = the zero-order Hankel transform of $p(\rho)$. The definition of the first-order transform follows from Eq. 4 if J_o changes to J_1, i.e., to the first-order Bessel function. $\bar{\sigma}_z$ and $\bar{\bar{\tau}}_{rz}$ can be evaluated in terms of $\bar{\Phi}$ by means of the well known expressions, given in Sneddon (8).

By noticing that

$$\bar{p}(\xi) = \int_o^R p_o(1-\rho^2)\, r\, J_o(\xi r)\, dr = 2p_o \frac{J_2(\xi R)}{\xi^2} \quad \ldots \ldots (6)$$

Eqs. 4 and 5 reduce to a system of two algebraic equations from which the two integration constants, $A(\xi)$ and $B(\xi)$, are readily computed. The final expression for $\bar{\Phi}$ is

$$\bar{\Phi}(\xi, z) = 2p_o \frac{\sqrt{g}\,[q_1 \exp(-\xi s_2 z) - q_2 \exp(-\xi s_1 z)]}{(s_1 - s_2)(ac - g)} \frac{J_2(\xi R)}{\xi^5} \quad \ldots \ldots (7)$$

in which $q_i = 1 - a s_i^2$, and $i = 1, 2$.

It is now evident that $\bar{\sigma}_z$ and $\bar{\bar{\tau}}_{rz}$ can be expressed as functions of ξ and z only. Similarly, the Hankel transforms of all the other stress and displacement components, $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{w}$ and $\bar{\bar{u}}$, can also be evaluated in terms of ξ and z by considering the appropriate transforms of both sides of Eq. 1 and using Eq. 7. To compute stresses and displacements as functions of z and r it is then sufficient to apply the inverse Hankel transformations. An example:

$$w(r, z) = \int_o^\infty \xi\, \bar{w}(\xi, z)\, J_o(\xi r)\, d\xi; \quad u(r, z) = \int_o^\infty \xi\, \bar{\bar{u}}(\xi, z)\, J_1(\xi r)\, d\xi \quad \ldots \ldots (8)$$

for the vertical and horizontal displacements, respectively, and similarly for the stress components. With the additional notation:

$$I_{mn}(x) = \int_o^\infty J_2(k)\, J_m(k\rho)\, k^{-n} e^{-kx}\, dk \quad \ldots \ldots (9)$$

evaluation of the Eq. 8 type integrals leads to the following final results:

$$w = \frac{2p_o R\sqrt{g}}{(s_1 - s_2)(ac - g)} [q_1 t_2 I_{02}(\lambda_2) - q_2 t_1 I_{02}(\lambda_1)] \quad \ldots \ldots (10)$$

$$u = -\frac{2p_o R\sqrt{g}\left(\dfrac{a}{G_{VH}} + f\right)}{(s_1 - s_2)(ac - g)} [q_1 s_2 I_{12}(\lambda_2) - q_2 s_1 I_{12}(\lambda_1)] \quad \ldots \ldots (11)$$

$$\sigma_z = \frac{2P_o}{s_1 - s_2} [s_1 I_{01}(\lambda_2) - s_2 I_{01}(\lambda_1)] \quad \ldots \ldots (12)$$

$$\sigma_r = \frac{2p_o}{(s_1 - s_2)\sqrt{g}} \left[s_1 I_{01}(\lambda_1) - s_2 I_{01}(\lambda_2) + \frac{\nu_{HH} - 1}{\rho} [s_1 q_2 I_{12}(\lambda_1) - s_2 q_1 I_{12}(\lambda_2)] \right] \quad \ldots \ldots (13)$$

$$\tau_{rz} = \frac{2p_o}{(s_1 - s_2)\sqrt{g}} [I_{11}(\lambda_1) - I_{11}(\lambda_2)] \quad \ldots \ldots (14)$$

in which $\lambda = \frac{z}{R}; \quad \lambda_i = \lambda s_i \quad (i = 1, 2)$ . . . . (15)

and a, b, c, g, f, and t_i $(i = 1, 2)$ = material constants given in Appendix I. The numerical evaluation of the so-called Lipschitz-Hankel type integrals, I_{mn} (Eq. 9), has been analyzed in Ref. 1. At the surface, $z = o$, and along the vertical Oz-axis, $\rho = o$, these integrals can be evaluated analytically and the resulting simple expressions for u and w at $z = o$, and σ_z, σ_r, and τ_{rz} at $\rho = o$ have been presented in a companion paper by the writer (2).

General Solution for Adhesive Parabolic Loading

Even the most flexible (in bending) raft foundations possess a significant in-plane stiffness. Thus, horizontal shear tractions are generated at the contact surface in response to the tendency of the soil surface to move horizontally (e.g., Eq. 11). If no slippage occurs between foundation and soil ('adhesive' contact), then no lateral displacements of the soil surface take place within the loaded area. Note that Schiffman (7), Hooper (4), and Keer (5) studied the response of an elastic isotropic halfspace to axisymmetric uniform, parabolic, and rigid-punch-type of 'adhesive' loading, respectively. In this note, the effect of 'adhesiveness' is evaluated for a cross-anisotropic, parabolically loaded medium.

In order to obtain a solution for this case, where no horizontal surface displacements are permitted at the soil foundation interface, it is sufficient to superpose stress and displacement components resulting from the 'frictionless' parabolic loading (Eqs. 10–14) with those generated from the following boundary conditions:

$$u^*(\rho) = \frac{p_o R}{4} \frac{\sqrt{g}\left(\frac{a}{G_{VH}} + f\right)(1 + a s_1 s_2)}{(ac - g)} \rho(2 - \rho^2) \quad o \le \rho \le 1 \quad . . . (16)$$

$$\sigma_z^*(\rho) = o \quad o \le \rho \le \infty \quad . . . . (17)$$

$$\tau_{rz}^*(\rho) = o \quad 1 < \rho < \infty \quad . . . . (18)$$

u^* (ρ) is equal in magnitude but opposite in sign to $u(\rho, o)$, obtained from Eq. 11 for $z = o$, $o \le \rho \le 1$ (see also Eq. 9*b* of the companion paper (2)). These new boundary conditions are of a mixed nature, i.e., unlike Eqs. 8 of Ref. 2, both stresses and displacements are prescribed at the boundary. Following similar steps with those described in the preceding section for 'frictionless' loading, the problem reduces to a set of dual-integral equations of the type solved by Busbridge (see Ref. 8). The final results of the analysis are given here for w^* (r, z) and σ^* (r, z) only:

$$w^* = \frac{2\sqrt{2}}{3\sqrt{\pi}} p_o R \frac{1 + a s_1 s_2}{\sqrt{g}\, s_1^2 s_2^2 (s_1^2 - s_2^2)(ac - g)} [s_2 y_2 t_1 K_o(\lambda_1)$$

$$- s_1 y_1 t_2 K_o(\lambda_2)] \quad . . . . (19)$$

$$\sigma_z^* = \frac{2\sqrt{2}\, p_o}{3\sqrt{\pi}} \frac{1 + a s_1 s_2}{\sqrt{g}\, (s_1^2 - s_2^2)(ac - g)} [y_2 \bar{y}_1 K_2(\lambda_1)$$

$$- y_1 \bar{y}_2 K_2(\lambda_2)] \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (20)$$

in which $y_i = g s_i^2 - 1, \quad \bar{y}_i = c - g s_i^2, \quad t_i = \dfrac{1}{G_{VH}} + f s_i^2; \quad i = 1, 2 \ldots \ldots$ (21)

$$K_n(x) = 4 \int_0^\infty J_{5/2}(k) J_0(k\rho) e^{-kx} k^{n-3/2} dk$$

$$+ \int_0^\infty J_{3/2}(k) J_0(k\rho) e^{-kx} k^{n-1/2} dk \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (22)$$

K_n is numerically evaluated using the results of Ref. 1, and the full solution

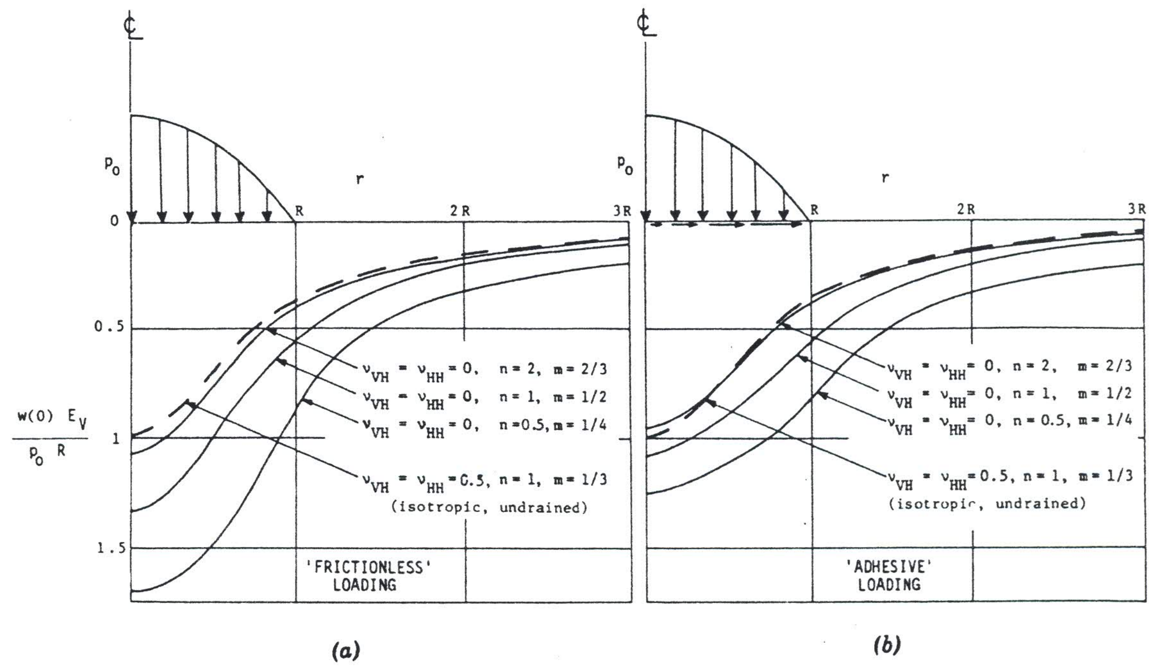

FIG. 1.—Surface Settlement Profile of Cross-Anisotropic Halfspace: (*a*) Frictionless Loading; (*b*) Adhesive Loading

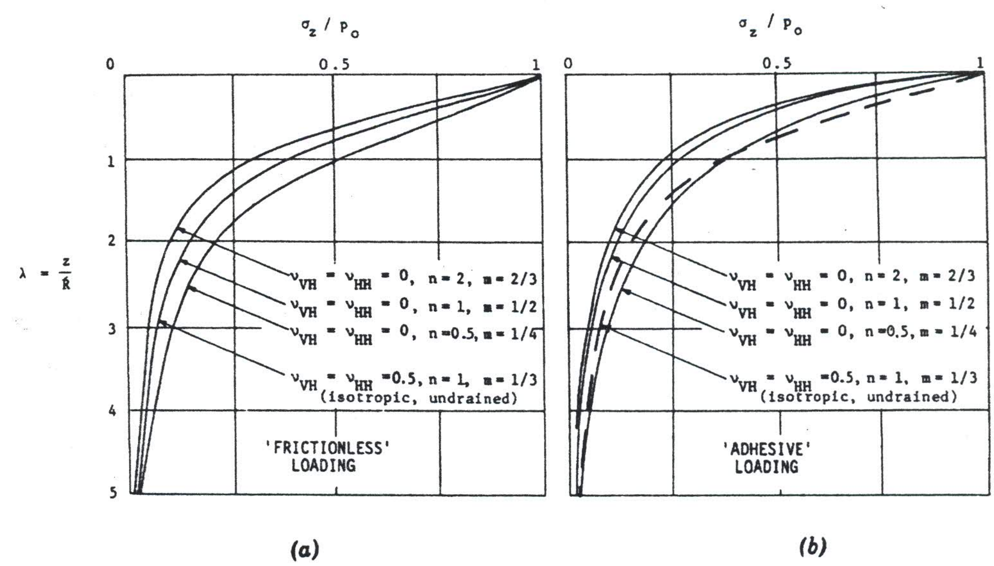

FIG. 2.—Distribution of Vertical Normal Stress Under Center of Load ($r = 0$): (*a*) Frictionless Loading; (*b*) Adhesive Loading

to the parabolic 'adhesive' loading problem may then be obtained by superimposing the solutions for parabolic 'frictionless' loading (Eqs. 10–14), and for specified horizontal displacements given by Eq. 16 (Eqs. 19 and 20).

Results and Conclusions

Figs. 1–3 show the effect of soil cross-anisotropy on surface settlement, $w(\rho)$, vertical stress along the central axis, $\sigma_z(\lambda)$, and radial stress across the surface, $\sigma_r(\rho)$, under both 'frictionless' and 'adhesive' loading conditions, for a material with zero Poisson's ratios. Such a medium may adequately represent actual soils loaded under drained conditions [e.g., London clay (3)]. Also shown for comparison in these figures (in dotted lines) is the response of an incompressible (i.e., undrained) isotropic medium.

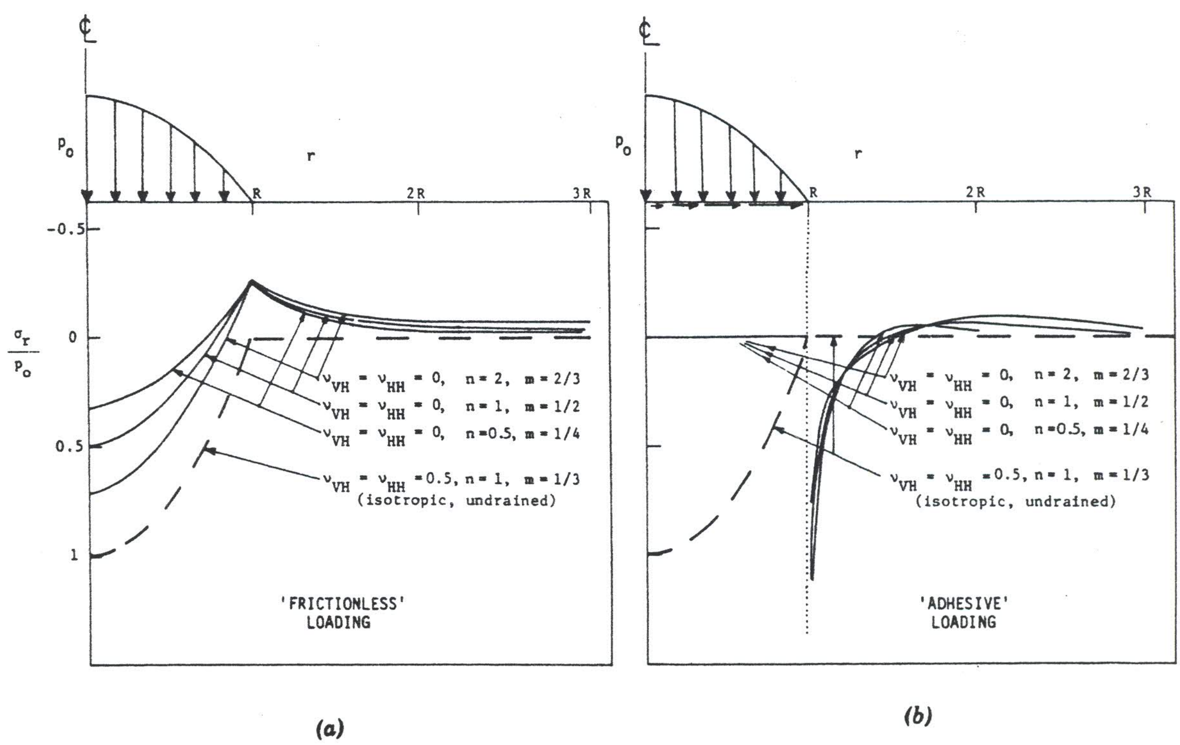

FIG. 3.—Distribution of Radial Normal Stress along Surface ($z = 0$): (a) Frictionless Loading; (b) Adhesive Loading

It is immediately apparent that for undrained soils the solutions for 'adhesive' and 'frictionless' contact coincide. This is hardly surprising in view of the fact that (any) undrained 'frictionless' normal load on a halfspace—isotropic or anisotropic—produces no radial horizontal displacements at the surface (2). However, in drained soils, interfacial shear tractions develop between the soil and a rough flexible footing as the latter prevents horizontal movements of the soil in the contact area. As a result, relative to the corresponding 'frictionless' loading, 'adhesive' loading of both isotropic and anisotropic media: (1) Substantially reduces both total and differential settlements within the loaded region; (2) reduces the concentration of vertical stresses, σ_z, along the central axis at shallow depths; and (3) changes drastically the distribution along the r-axis of the radial surface stresses, σ_r. Notice, nevertheless, that these differences between the two types of loading diminish at short distances away from the contact area, especailly in soils with small values of the n ratio. This is a consequence of the self-equilibrating nature of the shear tractions imposed by the rough foundation, in accordance with Saint-Venant's principle.

Finally, it is worth emphasizing that the two types of loading considered here ('adhesive' and 'frictionless') represent extreme cases of the possible mechanical behavior of the soil-load interface. A more realistic assumption is that of a contact obeying Coulomb's friction law. Such an interface would allow some slippage to occur thus, leading to stresses and displacements in between those of the two extreme cases.

Acknowledgment

Financial support by the National Science Foundation is kindly acknowledged.

Appendix I.—Soil Parameters $a, b, c, g,$ and f, s_1, s_2, t_i

When $n = E_H/E_V$, $m = G_{VH}/E_V$, and $j = n\nu_{VH}^2 - 1$, then a, b, c, g, and f are:

$$a = \nu_{VH}\frac{1+\nu_{HH}}{j}; \quad b = \frac{\left[n\nu_{VH}\left(\frac{1}{m} - V_{VH}\right) - \nu_{HH}\right]}{j};$$

$$c = a - \frac{1}{mj}; \quad g = \frac{\nu_{HH}^2 - 1}{nj}; \quad f = \frac{a(1-\nu_{HH} - 2n\nu_{VH}^2)}{nE_V\nu_{VH}};$$

$$t_i = \frac{1}{G_{VH}} + fs_i^2; \quad s_{1,2} = \left\{\frac{a + c \pm [(a+c)^2 - 4g]^{1/2}}{2g}\right\}^{1/2} \quad \ldots\ldots\ldots\ldots (23)$$

Appendix II.—References

1. Eason, G., et al., "On Certain Integrals of Lipschitz-Hankel Type Involving Products of Bessel Functions," *Philosophical Transactions of the Royal Society of London*, Series A247, No. 935, 1955, pp. 529–551.
2. Gazetas, G., "Stresses and Displacements in Anisotropic Soils," *Journal of the Geotechnical Engineering Division*, ASCE, Vol. 108, No. GT4, Proc. Paper 16998, Apr., 1982, pp. 532–553.
3. Hooper, J. A., "Elastic Settlement of a Circular Raft in Adhesive Contact with a Transversely Isotropic Medium," *Geótechnique*, Vol. 25, No. 4, 1975, pp. 691–711.
4. Hooper, J. A., "Parabolic Adhesive Loading of a Flexible Raft Foundation," *Geótechnique*, Vol. 26, No. 3, 1976, pp. 511–525.
5. Keer, L. M., "Mixed Boundary-Value Problems for an Elastic Half-space," *Proceedings*, Cambridge Philosophical Society, Vol. 63, 1967, pp. 1379–1386.
6. Lekhnitskii, S. G., *Theory of Elasticity of an Anisotropic Elastic Body*, Holden-Day, San Francisco, Calif., 1963.
7. Schiffman, R. L., "The Influence of Adhesion on Stresses and Displacements in an Elastic Half-Space," *Highway Research Record*, No. 282, pp. 213–219.
8. Sneddon, I. N., *Fourier Transforms*, McGraw-Hill Book Co., New York, N.Y., 1951.